The Mathematical way to Decide

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Decision making is the difficult part; it is based on a complex search for information, uncertainty, conflicting requirements and individual's personal preference. When the decision is based on a single attribute and the attribute is tangible, measurable, it actually is no decision as the decision is implied in the measurement. But when there are multiple attributes, criteria, functions etc., then there is confusion, and without adopting a logical procedure, wrong decisions may be taken. Over and above that, if at every stage the decision is questioned, which is true for all public officials, life can be difficult. This will slow down the decision process, leading to procrastination which is the worst enemy of the public official.

With multiple attributes or criteria, a more structured approach is required. Let us consider a few examples of decision making with multiple attributes,

1. Choosing a house – location, cost, floor area, amenities, distance from airport, independent
2. Choosing a company – salary, city, company reputation, working hours, growth opportunity
3. Choosing a car – style, mileage, type of transmission, type of fuel, comfort, cost, service
4. Selecting a supplier or contractor – quality, delivery, price, service, reputation
5. Selecting the best employee – loyalty, intelligence, attitude, special assignment
6. Selecting the best course – interest, employment opportunity, college, duration
7. Choosing a laptop or Smartphone, and so many more examples.

In all these examples, there are conflicting requirements or demands, and the decision maker does prioritize some attribute over the other. The ordinary decision maker does attach priority to the attributes and gives more weight to it. But there is a mathematical way to resolve this dilemma, called **Analytic Hierarchy Process**.

Analytic Hierarchy Process, **AHP method** developed by Thomas L. Saaty in the late 70's is perhaps, the most widely used decision making approach. Its validity is based on several applications by NASA, Xerox, General Motors and many other organizations..

AHP Steps

1. Model the decision problem by breaking it down into a hierarchy of interrelated decision elements; decision criteria, decision alternatives

2. Develop judgmental preferences of the decision alternatives for each criterion and judgmental importance of the decision criteria by pairwise comparisons

3. Compute relative priorities for each of the decision elements through a set of numerical calculations

4. Aggregate the relative priorities to arrive at a priority ranking of the decision alternatives

Fig 1: A typical decision hierarchy

We shall illustrate this method by a very simple example of choosing a car from 3 models with 3 criteria.

Model A, B and C are the car models.

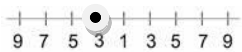
The three criteria for selection are price, mileage, comfort.

Attributes of the model should be tabulated. The comfort is adjudged by the user on a scale of 10 to quantify it.

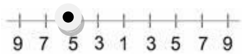
Model/Attribute	Price in Rs lakh	Mileage in km/l of fuel	Comfort on a scale of 10
A	850000	12	6
B	900000	11	8
C	1000000	9	10

It is certain that the three criteria will have different weights or priority for different people. Now, pair wise comparison has to be made for the criteria. Pair wise comparison is comparing the criterion to another for it respective importance vis-à-vis the other.

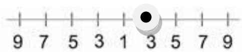
This scale may explain it better.

Price


Mileage

Price


Comfort

Mileage


Comfort

Points 1 to 9 are explained in the table,

1	Equal
2	Between Equal and Moderate
3	Moderate
4	Between Moderate and Strong
5	Strong
6	Between Strong and Very Strong
7	Very Strong
8	Between Very Strong and Extreme
9	Extreme

So, if you mark 3 towards the price on the pair wise comparison of price and mileage, it would mean price is preferred moderately over mileage.

Next step is to form a matrix of the pair wise comparisons.

	Price	Mileage	Comfort
Price	1	3	5
Mileage	1/3	1	1/3
Comfort	1/5	3	1

Let's see how this matrix is formed. Since we have three attributes to be compared a 3 by 3 matrix is formed. The diagonal elements of the matrix are obviously 1. The upper row has to be filled. The rules are:

1. *If the attribute is more preferred i.e. judgment value is on the **left** side of 1, we put the **actual judgment** value.*
2. *If the attribute is less preferred i.e. judgment value is on the **right** side of 1, we put the **reciprocal** value.*

In this case, the attribute, ***price*** is 3 points higher than ***mileage*** and 5 points higher for ***comfort***. To fill the lower triangular matrix, we use the reciprocal values of the comparisons.

Without going into the mathematics of matrix, the AHP process will be explained to let the reader have a fair idea of this wonderful technique, which has established itself as the prime decision making technique in management science.

We sum each column of the matrix to get

	Price	Mileage	Comfort
Price	1	3	5
Mileage	1/3	1	1/3
Comfort	1/5	3	1
Sum	23/15	7	19/3

Now we divide each element of the matrix with the sum of its column, to get the normalized relative weight. The sum of each column should be 1.

	Price	Mileage	Comfort
Price	15/23	3/7	15/19
Mileage	5/23	1/7	1/19
Comfort	3/23	3/7	3/19
Sum	1	1	1

The normalized principal Eigen vector (a term used with matrices) is obtained by averaging the rows as shown (all fractions converted to decimal numbers),

	Price	Mileage	Comfort	**Average or Priority Vector**
Price	0.6522	0.4286	0.7895	**0.6234**
Mileage	0.2174	0.1429	0.0526	**0.1376**
Comfort	0.1304	0.4286	0.1579	**0.2390**
Sum	1	1	1	**1**

This normalized principal Eigen vector is also called **priority vector**. The priority vector shows relative weights among the attributes that are to be compared. For the choice of car, Price has 62.34% weight, Mileage 13.76% and Comfort 23.9%.

But an important point here, how consistent the comparison is. For the consistency check, we have to calculate the Principal Eigen value. Principal Eigen value is obtained from the summation of products between each element of Eigen vector and the sum of columns of the reciprocal matrix.

	Price	Mileage	Comfort	Priority Vector
Price (P)	1	3	5	**0.6234**
Mileage (M)	1/3	1	1/3	**0.1376**
Comfort (C)	1/5	3	1	**0.2390**
Sum	**23/15**	**7**	**19/3**	
Calculation of Principal Eigen Vector	**23/15 x 0.6234**	**7 x 0.1376**	**19/3 x 0.2390**	**Adding up 3.43**

What is the need for this computation and the meaning of consistency?

Subjective judgment is subjected to several inconsistencies and there is a method in AHP to assess this. In this example in the first row, Price(P) is compared to Mileage (M) and Comfort (C) and P is placed at higher importance level 5 with respect to C, as compared to 3 for M. It is evident that M is more important than C.

But in the second row, M has been shown to be lower in preference to C, i.e. 1/3. This does seem inconsistent. Let's check this mathematically.

With AHP, there is a way to measure the degree of consistency; and if unacceptable, the pairwise comparisons may be revised.

Saaty gave a measure of consistency, called Consistency Index as deviation or degree of consistency using the following formula

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

Thus in our previous example, we have $\boldsymbol{\lambda_{max}}$ is the principal eigen vector, which was calculated to be 3.43. There are three comparisons, $\boldsymbol{n}$=3, thus the consistency index is

$$CI = \frac{\lambda_{max} - n}{n - 1} = \frac{3.43 - 3}{3 - 1}$$

=0.215

This index is compared with a Random Consistency Index (RI), which was estimated by Saaty. This index depends on the number of attributes denoted by $\boldsymbol{n}$.

Random Consistency Index

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

Using RI the Consistency Ratio (CR) is found out, which must be less than 10% for decision to be termed consistent. CR is a comparison between Consistency Index and Random Consistency Index, or in formula

$$CR = \frac{CI}{RI}$$

If the value of Consistency Ratio is smaller or equal to 10%, the inconsistency is acceptable. If the Consistency Ratio is greater than 10%, we need to revise the subjective judgment.

Let us check this for the car example, ***CI***=0.215

Here, ***n***=3, thus ***RI***=0.58 and ***CR***=0.215/0.58=37%

This is much higher than 10%, thus an inconsistent decision. We will have to revise the comparison. This inconsistency was discussed earlier and was easily detected as the matrix is small; if there is large number of attributes then inconsistencies may emerge and can be detected only by using this index. Thus AHP is a self correcting system, which is its main advantage and reason for wide acceptability.

Let us now revise the comparisons,

	Price	Mileage	Comfort
Price	1	3	5
Mileage	1/3	1	**3**
Comfort	1/5	**1/3**	1
Sum	23/15	13/3	9

Mileage vs comfort has been revised and mileage has been placed at higher preference, repeating all steps as done before, the final matrix and principal eigen vector calculations are,

	Price	Mileage	Comfort	Priority Vector
Price	0.65217	0.69231	0.55556	**0.6333**
Mileage	0.21739	0.23077	0.33333	**0.2605**
Comfort	0.13043	0.07692	0.11111	**0.1062**
Sum	**1.5333**	**4.3333**	**9.0000**	λ_{max}= **3.055**

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

Thus, ***CI***=0.028 and ***CR***=0.028/0.58=4.8%

which proves that this is a consistent comparison.

Finally the choice of car will depend on Price with 63.33%, Mileage 26.05% and Comfort 10.62% weight.

A very simple method of finding the best alternative from the available models of car is to find the weighted sum for each alternative and the one with highest score is the choice.

Model/Attribute	**Price in Rs lakh**	**Mileage in km/l of fuel**	**Comfort on a scale of 10**
A	850000	12	6
B	900000	11	8
C	1000000	9	10

To use the weights on this matrix, the attributes have to be normalized based on type of attribute, higher the better or lower the better. In the present case, Price should be lowest, mileage and comfort highest.

Rules for normalization are

1. For higher the better, i.e. positive criteria, take the highest value as denominator and divide all elements of the column by it.
2. For lower the better, i.e. negative criteria, take the highest value as numerator and element as denominator for each cell of the respective column.

Thus the normalized table is

	Price		Mileage		Comfort	
A	1.18	(1000000/850000)	1.00	(12/12)	0.60	(6/10)
B	1.11	(1000000/900000)	0.92	(11/12)	0.80	(8/10)
C	1.00	(1000000/1000000)	0.75	(9/12)	1.00	(10/10)

Using the weights computed from AHP, the score for each car will is calculated and cars ranked in that order.

The weight of each criterion is multiplied with the cell value of row for each alternative and all these values are added up to get the final score. The alternative with the highest score is the choice.

	Price	Mileage	Comfort	Total Score	
	Weights				
	0.6333	**0.2605**	**0.1062**		
A	0.7451	0.2605	0.0637	**1.0693**	BEST
B	0.7037	0.2388	0.0850	1.0274	
C	0.6333	0.1954	0.1062	0.9349	

{Example of calculation for A: (1.18x0.6333)+(1x0.2605)+(0.60x0.1062)=1.0693}

Thus in this case, car A is found to be the best option.

This example is very simple, it may seem that so many calculations were not needed. Imagine a situation when you have 10 bidders to be qualified on the basis of 15 attributes. The matrix will have 150 cells. The important thing is to understand the method, excel sheets can be used for all the calculations and there are software packages available for AHP etc.

There are many more such methods which are designated as Multi Criteria Decision Making (MCDM) techniques. Few of the popular ones are,

1. SMART (Simple Multi Attribute Rating Technique)
2. TOPSIS (Technique for Order Preference by Similarity to the Ideal Solution)
3. ELECTRE (ELimination Et Choix Traduisant la REalité; ELimination and Choice Expressing Reality)
4. ANP (Analytic Network Process)

The technique can be applied to contracts and several other situations when there are conflicting choices and too many of them. This is a very mathematical way to decide and almost without any discretion, everything is quantified.

Suggested Reading and Internet resources

Saaty, T.L., 1980. “The Analytic Hierarchy Process.” McGraw-Hill, New York.

Saaty, T.L., 2001. “Decision Making for Leaders: The Analytic Hierarchy Process for Decisions in a Complex World, (Analytic Hierarchy Process Series, Vol. 2)”, RWS Publication, Pittsburgh.

Bhushan, N., Rai, K., 2004. “Strategic Decision Making- Applying the Analytic Hierarchy Process, Series: Decision Engineering”, Springer India.

http://mi.boku.ac.at/ahp/ahptutorial.pdf

http://www.business.pitt.edu/faculty/papers/saaty-into-to-ahp-mathematics.pdf

http://www.dii.unisi.it/~mocenni/Note_AHP.pdf

http://analytics.ncsu.edu/sesug/2012/SD-04.pdf

http://giswin.geo.tsukuba.ac.jp/sis/gis_seminar/How%20to%20do%20AHP%20analysis%20in%20Excel.pdf

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